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NUMERICAL EVALUATION BY HARMONIC ANALYSIS OF THE ϵ-FUNCTION OF THE THEODORSEN ARBITRARY-AIRFOIL POTENTIAL THEORY

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SUMMARY

A finite trigonometric series is fitted by harmonic analysis as an approximation function to the ψ-function of the Theodorsen arbitrary-airfoil potential theory. By harmonic synthesis the corresponding conjugate trigonometric series is used as an approximation to the ϵ-function. A set of coefficients of particularly simple form is obtained algebraically for direct calculation of the ϵ-values from the corresponding set of ψ-values. The formula for ϵ is

$$\epsilon(\varphi) = \sum_{k=1}^{n} \frac{1}{n} \cot \frac{k\pi}{2n} \left(\psi_{-k} - \psi_{k}\right)$$

where the summation is for odd values of k only and

$$\psi_k \equiv \psi\left(\varphi + \frac{k\pi}{n}\right)$$

INTRODUCTION

In the determination of the flow about an arbitrary airfoil (references 1 and 2) the problem arises of transforming a curve, nearly circular, into a circle. This transformation, a basic problem in conformal mapping, is further reduced to the determination of the following two conjugate Fourier series:

$$\left.\begin{aligned}\psi &\equiv a_0 + \sum_{m=1}^{\infty} \left(a_m \cos m\varphi + b_m \sin m\varphi\right) \\ \epsilon &\equiv \sum_{m=1}^{\infty} \left(a_m \sin m\varphi - b_m \cos m\varphi\right)\end{aligned}\right\} \qquad (1)$$

(See references 1 and 2 for significance of notation.) The following integral relations are equivalent to series relations (1):

$$\left.\begin{aligned}\psi(\varphi) &= \frac{1}{2\pi}\int_0^{2\pi} \psi(\varphi')d\varphi' + \frac{1}{2\pi}\int_0^{2\pi} \epsilon(\varphi') \cot \frac{\varphi' - \varphi}{2} d\varphi' \\ \epsilon(\varphi) &= -\frac{1}{2\pi}\int_0^{2\pi} \psi(\varphi') \cot \frac{\varphi' - \varphi}{2} d\varphi'\end{aligned}\right\} \qquad (2)$$

It is convenient to introduce a new variable $s = \varphi' - \varphi$ in relations (2). Because of the cyclical nature of these functions, the limits of integration may be written $-\pi$ to π. When the integral is broken into two parts, $-\pi$ to 0 and 0 to π, and $-s$ is substituted for s in the first part, the following relations are obtained:

$$\left.\begin{aligned}\psi(\varphi) &= \frac{1}{2\pi}\int_0^{2\pi} \psi(\varphi)d\varphi + \frac{1}{2\pi}\int_0^{\pi} \left[\epsilon(\varphi + s) - \epsilon(\varphi - s)\right] \cot \frac{s}{2} ds \\ \epsilon(\varphi) &= \frac{1}{2\pi}\int_0^{\pi} \left[\psi(\varphi - s) - \psi(\varphi + s)\right] \cot \frac{s}{2} ds\end{aligned}\right\} \qquad (3)$$

Thus, by use of relations (1), (2), or (3), ϵ may be determined if ψ is known or ψ may be determined if ϵ is known.

In the airfoil problem ψ is specified as a function[1] of φ by means of a curve and ϵ is to be determined. In theory the Fourier coefficients may be determined in relations (1) but in practice, because of the unknown analytic nature of the curve, it is necessary to resort to some type of numerical approximation.

In references 1 and 2 an approximate method of handling the integrals of relations (2) is presented. In reference 3 a refinement of this method is given for the same integrals. An alternative procedure is to approximate relations (1) by a finite trigonometric series and then to determine the coefficients by harmonic analysis. A development of this method is now given.

HARMONIC ANALYSIS

The ψ-function is to be approximated by a finite trigonometric series given by

$$\psi(\varphi) = A_0 + A_1 \cos \varphi + \ldots + A_{n-1} \cos (n-1)\varphi + A_n \cos n\varphi$$
$$+ B_1 \sin \varphi + \ldots + B_{n-1} \sin (n-1)\varphi$$
$$= A_0 + \sum_{m=1}^{n-1} \left(A_m \cos m\varphi + B_m \sin m\varphi\right) + A_n \cos n\varphi$$

If ψ is specified at $2n$ equally spaced intervals in the range $0 \leqq \varphi \leqq 2\pi$ - that is, $0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{(2n-1)\pi}{n}$ - then

$$A_0 = \frac{1}{2n} \sum_{r=0}^{2n-1} \psi_r$$

[1]In practice, ψ is given as a function of $\theta = \varphi - \epsilon$ and therefore ψ is taken as a function of φ as a first approximation. An iteration process is necessary to determine both ψ and ϵ correctly as functions of φ.

where $\psi_r = \psi\left(\frac{r\pi}{n}\right)$

$$A_m = \frac{1}{n}\sum_{r=0}^{2n-1} \psi_r \cos m\frac{r\pi}{n}$$

$$B_m = \frac{1}{n}\sum_{r=0}^{n-1} \psi_r \sin m\frac{r\pi}{n}$$

$$A_n = \frac{1}{2n}\sum_{r=0}^{2n-1} (-1)^r \psi_r$$

$$B_n = 0$$

Now

$$\psi(\theta) = \sum_{m=1}^{n-1}\left(A_m \sin m\theta - B_m \cos m\theta\right) + A_n \sin n\theta$$

$$= \frac{1}{n}\sum_{m}\left(\sin m\theta\sum_{r} \psi_r \cos m\frac{r\pi}{n} - \cos m\theta\sum_{r} \psi_r \sin m\frac{r\pi}{n}\right)$$

$$+ \frac{1}{2n}\sin n\theta\sum_{r}(-1)^r \psi_r$$

$$= \frac{1}{n}\sum_{m}\sum_{r} \psi_r \sin m\left(\theta - \frac{r\pi}{n}\right) + \frac{1}{2n}\sin n\theta\sum_{r}(-1)^r \psi_r$$

When the order of summation is interchanged,

$$\epsilon(\varphi) = \frac{1}{n}\sum_{r=0}^{2n-1}\psi_r\sum_{m=1}^{n-1}\sin m\left(\varphi - \frac{r\pi}{n}\right) + \frac{1}{2n}\sin n\varphi\sum_{r=0}^{2n-1}(-1)^r\psi_r$$

Now if ϵ is evaluated at the same points φ at which the values of ψ are given, that is, at the points $\varphi = \frac{r'\pi}{n}$, the variable becomes

$$\varphi - \frac{r\pi}{n} = \frac{(r' - r)\pi}{n} = -\frac{k\pi}{n}$$

and the last term becomes zero. After this substitution is performed,

$$\psi_r = \psi\left(\frac{r\pi}{n}\right) = \psi\left(\frac{r'\pi}{n} + \frac{k\pi}{n}\right) = \psi\left(\varphi + \frac{k\pi}{n}\right) \equiv \psi_k$$

and

$$\epsilon(\varphi) = -\frac{1}{n}\sum_{k=0}^{2n-1}\psi_k\sum_{m=1}^{n-1}\sin m\frac{k\pi}{n}$$

The summation with respect to k is also from 0 to $2n - 1$ because of the periodicity. But when k is odd,

$$\sum_{m=1}^{n-1}\sin m\frac{k\pi}{n} = \cot\frac{k\pi}{2n}$$

and when k is even,

$$\sum_{m=1}^{n-1}\sin m\frac{k\pi}{n} = 0$$

For odd values of k, therefore

$$\epsilon(\varphi) = -\frac{1}{n}\sum_{k=1}^{2n-1}\psi_k\cot\frac{k\pi}{2n}$$

or

$$\epsilon(\varphi) = \frac{1}{n}\sum_{k=1}^{n}\left(\psi_{-k} - \psi_{k}\right) \cot \frac{k\pi}{2n}$$

Finally, then,

$$\epsilon(\varphi) = \sum_{k=1}^{n} C_k\left(\psi_{-k} - \psi_{k}\right) \tag{4}$$

where

$$\psi_k \equiv \psi\left(\varphi + \frac{k\pi}{n}\right)$$

and for odd values of k

$$C_k \equiv \frac{1}{n} \cot \frac{k\pi}{2n}$$

and for even values of k

$$C_k \equiv 0$$

Equation (4) thus gives the same result as is obtained by harmonic analysis and synthesis. Comparison with equations (3) indicates that equation (4) may also be interpreted as the evaluation of this integral by the ordinary rectangular summation formula using intervals of width $2\pi/n$ and using the value of the integrand at the midpoint of each interval, that is, at $s = \frac{k\pi}{n}$ where k is odd.

PRACTICAL OBSERVATIONS

Equation (4) uses only one-half the available information. It is evident that all the points may be used because all the given n points may be considered as alternate (odd) points of a system of 2n points. The

value of ϵ so computed is, of course, to be plotted at φ-points midway between the given ψ-points. The n values of ψ therefore give values of ϵ corresponding to an approximation function consisting of a trigonometric series of n - 1 harmonics. Values of the coefficient C_k for n = 10, 20, 40, and 80 are given in table I. For smooth curves the present method for n = 20 is more accurate than the 40-point method of reference 3 and requires only one-half as much computational work.

How to handle small irregularities or bumps in the ψ-curve is of interest. One procedure is to fair through the bump and to designate the faired curve $\overline{\psi}$. The deviation from $\overline{\psi}$ is a $\Delta\psi$-curve. The conjugate $\overline{\epsilon}$ may be determined in the usual manner and a conjugate $\Delta\epsilon$ may be determined by use of a very small interval, say, n = 200. The desired ϵ-values are given by the sum of $\overline{\epsilon}$ and $\Delta\epsilon$. This method cannot be justified on strict mathematical grounds but is probably more than adequate for engineering purposes.

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2. Theodorsen, T., and Garrick, I. E.: General Potential Theory of Arbitrary Wing Sections. NACA Rep. No. 452, 1933.

3. Naiman, Irven: Numerical Evaluation of the ϵ-Integral Occurring in the Theodorsen Arbitrary Airfoil Potential Theory. NACA ARR No. L4D27a, 1944.

TABLE I.- VALUES OF C_k FOR USE WITH EQUATION (4)

k	C_k			
	n = 10	n = 20	n = 40	n = 80
1	0.63138	0.63531	0.63629	0.63654
3	.19626	.20827	.21122	.21196
5	.10000	.12071	.12568	.12691
7	.05095	.08159	.08864	.09037
9	.01584	.05854	.06777	.06999
11		.04270	.05423	.05697
13		.03064	.04464	.04790
15		.02071	.03742	.04121
17		.01200	.03171	.03605
19		.00394	.02704	.03194
21			.02311	.02858
23			.01971	.02577
25			.01670	.02339
27			.01400	.02133
29			.01153	.01953
31			.00922	.01794
33			.00705	.01651
35			.00497	.01523
37			.00296	.01407
39			.00098	.01300
41				.01202
43				.01111
45				.01026
47				.00946
49				.00871
51				.00800
53				.00733
55				.00668
57				.00606
59				.00547
61				.00489
63				.00433
65				.00379
67				.00326
69				.00274
71				.00223
73				.00173
75				.00123
77				.00074
79				.00025

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